Int. J. Solids Structures, 1974, Vol. 10, pp. 461–468. Pergamon Press. Printed in Gt. Britain.

STRESS BOUNDS FOR TWISTED BARS OF STRIP CROSS SECTION

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(*Received* 23 *April* 1973; *revised* 4 *September* 1973)

Abstract—Bounds are established for the maximum resultant shear stress, and used to investigate the approximation involved in the value predicted for thin sections on the basis of the membrane analogy.

1. INTRODUCTION

In a recent investigation[1], we derived an upper bound on the maximum resultant shear stress as it is determined by the St. Venant torsion theory for homogeneous bars. Although it was concluded that the bound was optimal for the circular cross section, examples involving other cross sections led to instances in which the bound is undesirably crude. In effect, we were left with the task of seeking types of cross sections for which acceptable accuracy can be obtained.

The present paper is concerned with a modification of the bound given in [1], a modification appropriate to thin cross sections. The result is a bound that improves in accuracy as the thickness tends to zero. Since the exact value of the maximum resultant shear stress is not in general accessible for the class of thin strips under consideration, we have also deduced a lower bound for it, the idea being to assess the upper bound by means of its proximity to the lower bound.

For a bar of rectangular cross section subject to a twisting moment of magnitude m,

$$\sigma = \frac{3m}{a^2 b}$$

is regarded as a useful approximation to the resultant shear stress in the event the thickness to width ratio a/b is small.† This result is also considered useful if the cross section consists of a thin curved strip‡ of thickness a and developed length b. The bounds in the present paper confirm this procedure, and provide a straightforward means for bounding the error committed by using it.

2. PRANDTL'S STRESS FUNCTION AND THE BASIC SCHEME FOR OBTAINING BOUNDS

Let $\mathscr{D}$ stand for the open, plane domain occupied by a cross section of the bar, and let (x, y, z) stand for cartesian coordinates relative to a frame that contains $\mathscr{D}$ in its (x, y) plane. If $\mathscr{D}$ is simply connected, the Prandtl stress function Φ is governed by

$$\nabla^2\Phi = -2 \quad \text{on} \quad \mathscr{D}, \qquad \Phi = 0 \quad \text{on} \quad \mathscr{C}, \tag{2.1§}$$

† See [2], p. 240.
‡ *Op. cit.* p. 244.
§ See p. 114 *et sec.* of [3].

where $\mathscr{C}$ designates the boundary of $\mathscr{D}$, and ∇^2 denotes the 2-dimensional Laplace operator. The only non-zero stresses that arise in the St. Venant torsion problem are the shear stresses t_x and t_y that act on planes normal to the z-axis. These are found in terms of Φ and the applied twisting moment M from

$$t_x = \frac{M}{k}\frac{\partial\Phi}{\partial y}, \qquad t_y = -\frac{M}{k}\frac{\partial\Phi}{\partial x} \tag{2.2†}$$

where

$$k = 2\int_{\mathscr{D}} \Phi \, \mathrm{d}A. \tag{2.3}$$

If μ is the shear modulus, then μk is the torsional rigidity.

We are interested in approximating the maximum magnitude of the shear stress,

$$t = \max_{\mathscr{R}} \sqrt{t_x^{\,2} + t_y^{\,2}}, \tag{2.4}$$

where $\mathscr{R} = \mathscr{D} \cup \mathscr{C}$. In view of (2.2),

$$t = \frac{m}{k}\,\tau, \tag{2.5}$$

provided

$$m = |M| \quad \text{and} \quad \tau = \max_{\mathscr{R}} |\nabla\Phi|. \tag{2.6}$$

Equation (2.5) shows that bounds on t can be found from bounds on τ and k. Indeed, if

$$0 < \underline{\tau} \le \tau \le \bar{\tau}, \qquad 0 < \underline{k} \le k \le \bar{k}, \tag{2.7}$$

(2.5) implies

$$\underline{\tau}/\bar{k} \le t/m \le \bar{\tau}/\underline{k}. \tag{2.8}$$

3. BOUNDS ON τ

In order to get an upper bound, we assumed in [1] that the curvature of $\mathscr{C}$ is continuous at all but a finite number of points. At these points the curvature is permitted to have a jump discontinuity of the kind appropriate to an exterior corner. Re-entrant corners are excluded. The convention was adopted that the curvature is positive or negative according to whether the center of curvature is on the inward or outward normal ray. Thus, the minimum curvature, κ, is negative if $\mathscr{R}$ is not convex, whereas it is positive if $\mathscr{R}$ is strictly convex. The upper bound derived in [1] is given by

$$\tau \le \rho(2 - \kappa\rho), \tag{3.1}$$

where ρ stands for the maximum radius attainable by disks contained in $\mathscr{R}$. This bound is optimal for a circular cross section, and if the cross section is elliptic, has the property

$$\frac{\tau}{\rho(2 - \kappa\rho)} \to 1 \quad \text{as} \quad e \to 1, \tag{3.2}$$

where e denotes the eccentricity.

† A number of results in this paper involve assuming that Φ is smooth. We henceforth assume that Φ is twice continuously differentiable on $\mathscr{D}$ and once so on its closure.

We turn now to the task of finding a lower bound for τ. Toward this end, we note that a simple application of the divergence theorem in conjunction with (2.1) yields

$$-2A = \int_{\mathscr{C}} \frac{\partial \Phi}{\partial n}\, ds,$$

where $\partial\Phi/\partial n$ designates the outward normal derivative, and A is the cross-sectional area. Thus, if l denotes the length of $\mathscr{C}$, then

$$\max_{\mathscr{C}} \left| \frac{\partial \Phi}{\partial n} \right| \geq \frac{2A}{l}. \tag{3.4}$$

Since Φ is constant on $\mathscr{C}$, it follows that

$$\left| \frac{\partial \Phi}{\partial n} \right| = |\nabla \Phi| \quad \text{on} \quad \mathscr{C}.$$

Hence (3.4) and (2.6) furnish

$$\tau \geq \frac{2A}{l}. \tag{3.5}$$

Just as in (3.1), equality holds in (3.5) if $\mathscr{C}$ is a circle.

4. BOUNDS ON k

It is pointed out in [4] that if $\boldsymbol{\psi}$ is a continuously differentiable vector field on $\mathscr{R}$, and

$$\nabla \cdot \boldsymbol{\psi} = -2 \quad \text{on} \quad \mathscr{D}, \tag{4.1}$$

then

$$k \leq \int_{\mathscr{D}} |\boldsymbol{\psi}|^2\, dA. \tag{4.2}$$

In the case of the rectangle

$$\mathscr{D} = \left\{ (x, y) \,\middle|\, -\frac{b}{2} \leq x \leq \frac{b}{2},\ -\frac{a}{2} \leq y \leq \frac{a}{2} \right\},$$

the choice

$$\psi_x = 0, \qquad \psi_y = -2y$$

leads to

$$k \leq \frac{ba^3}{3}. \tag{4.3}$$

It is well known[5] that

$$\frac{3k}{ba^3} \to 1 \quad \text{as} \quad a \to 0.$$

Thus, the right hand member in (4.3) not only bounds k from above, but also furnishes an approximation that improves in accuracy as the thickness tends to zero.

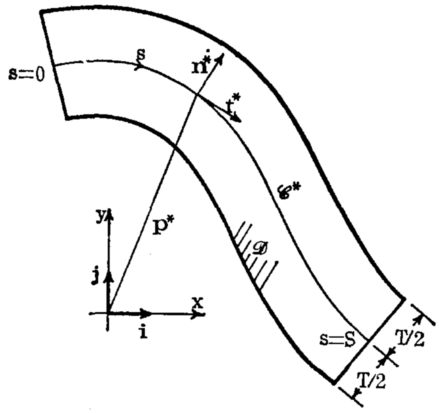

Fig. 1.

It would serve our purposes to be able to conclude that if $\mathscr{D}$ is a strip of developed length S and thickness T, then,

$$k \leq \frac{ST^3}{3}. \tag{4.4}$$

Although we have not been able to resolve this issue, our efforts in this direction have led to an adequate upper bound. Of concern is a domain of the type suggested by Fig. 1. In this figure, $\mathscr{C}^*$ represents a smooth† arc parametrized with respect to its length,

$$\mathscr{C}^* = \{(x, y) \,|\, \mathbf{p}^*(s) = x\mathbf{i} + y\mathbf{j}, \quad 0 \leq s \leq S\}. \tag{4.5}$$

The boundary of $\mathscr{D}$ consists of two arcs parallel to $\mathscr{C}^*$ at a distance $T/2$, and two straight line segments normal to $\mathscr{C}^*$ at its ends. We assume that $\mathbf{p}^*$ is twice continuously differentiable on $[0, S]$. The normal $\mathbf{n}^*$ and tangent $\mathbf{t}^*$ to $\mathscr{C}^*$ are given by

$$\mathbf{t}^* = \frac{\mathrm{d}}{\mathrm{d}s}\,\mathbf{p}^*, \qquad \mathbf{n}^* = \mathbf{k} \times \mathbf{t}^*, \tag{4.6}$$

where $\mathbf{k} = \mathbf{i} \times \mathbf{j}$. We also have the Frenet formulas

$$\frac{\mathrm{d}}{\mathrm{d}s}\,\mathbf{t}^* = \kappa^*\mathbf{n}^*, \qquad \frac{\mathrm{d}}{\mathrm{d}s}\,\mathbf{n}^* = -\kappa^*\mathbf{t}^*, \tag{4.7}$$

where χ^* designates the curvature of $\mathscr{C}^*$.

If we put

$$\mathbf{p}(s, \lambda) = \mathbf{p}^*(s) + \lambda\mathbf{n}^*(s), \qquad 0 \leq s \leq S, \qquad -T/2 \leq \lambda \leq T/2, \tag{4.8}$$

the curves $s =$ constant and $\lambda =$ constant serve as coordinate curves for an orthogonal system of curvilinear coordinates, as one may readily verify with the aid of (4.6), (4.7). Since $\mathscr{C}^*$ is smooth, the mapping $(s, \lambda) \to (x, y)$ defined by (4.8) is differentiable and one-to-one, provided either $\mathscr{D}$ is a rectangle (in which case $\kappa^*(s) = 0$, $0 \leq s \leq S$) or

$$T/2 < \min \frac{1}{|\kappa^*(s)|} \qquad (0 \leq s \leq S). \tag{4.9}$$

† See [6], p. 97.

We turn now to the task of getting an upper bound on k. Let

$$f(\eta, \lambda) = \begin{cases} \dfrac{1-\lambda\eta}{\eta} + \dfrac{T}{1-\lambda\eta} \dfrac{1}{\ln \dfrac{(1-T\eta/2)}{(1+T\eta/2)}} & (0 < |\eta| \leq \max|\kappa^*|, 0 \leq |\lambda| \leq T/2) \\ -2\lambda & (\eta = 0, 0 \leq |\lambda| \leq T/2). \end{cases} \tag{4.10}$$

It is important to be aware that f is continuously differentiable for $0 \leq |\lambda| \leq T/2$, $0 \leq |\eta| \leq \max|\kappa^*|$. This ensures that the vector field

$$\boldsymbol{\psi}(s, \lambda) = f(\kappa^*(s), \lambda)\mathbf{n}^*(s) \tag{4.11}$$

is defined and continuously differentiable for

$$0 \leq s \leq S, \qquad -T/2 \leq \lambda \leq T/2.$$

From [7] (pp. 104–107), one finds that

$$\nabla \cdot \boldsymbol{\psi}(s, \lambda) = \frac{1}{1-\lambda\kappa^*(s)} \frac{\partial}{\partial\lambda} [(1-\lambda\kappa^*(s))f(\kappa^*(s), \lambda)].$$

A brief calculation involving (4.10) gives

$$\nabla \cdot \boldsymbol{\psi} = -2.$$

We thus conclude that $\boldsymbol{\psi}$, defined by (4.11) and (4.10), is a candidate for (4.2). According to [7] (pp. 104–107), we may write

$$\int_{\mathscr{D}} |\boldsymbol{\psi}|^2 \, \mathrm{d}A = \int_0^S \int_{-T/2}^{T/2} |\boldsymbol{\psi}|^2 (1-\lambda\kappa^*(s)) \, \mathrm{d}\lambda \, \mathrm{d}s. \tag{4.12}$$

Consequently, and by (4.11), (4.10), (4.2), there follows

$$k \leq \int_0^S \left\{ \frac{1}{4(\kappa^*(s))^2} [4T + T^3(\kappa^*(s))^2] + \frac{T^2}{\kappa^*(s) \ln\left[\dfrac{1-T\kappa^*(s)/2}{1+T\kappa^*(s)/2}\right]} \right\} \mathrm{d}s. \tag{4.13†}$$

At this stage, there is little we can do in the way of evaluating the integral in (4.13) without specifying κ^*. Instead of pursuing this, we will aim for an upper bound on the integral. Toward this end, we expand $\ln(1+u)$ in a power series about $u = 0$:

$$\ln(1+u) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} u^k \qquad (|u| < 1).$$

† It is understood that the integrand in (4.13) has the value $T^3/3$ for $\kappa^*(s) = 0$ (recall 4.10).

This gives

$$u \ln\left(\frac{1-u}{1+u}\right) = \sum_{k=1}^{\infty} [(-1)^{2k+1} - (-1)^{k+1}] \frac{u^{k+1}}{k}$$

$$= -2u^2 - \tfrac{2}{3}u^4 + u^6 \sum_{k=5}^{\infty} [(-1)^{2k+1} - (-1)^{k+1}] \frac{u^{k-5}}{k}$$

$$\geq -2u^2 - \tfrac{2}{3}u^4 - \tfrac{2}{5}u^6 \sum_{k=0}^{\infty} |u|^k = -2u^2 - \tfrac{2}{3}u^4 - \frac{2u^6}{5(1-|u|)}$$

for $0 \leq |u| < 1$. Therefore, for $0 < |u| < \frac{1}{2}$,

$$\frac{1}{u \ln\left(\frac{1-u}{1+u}\right)} \leq -\frac{1}{2u^2 + \frac{2}{3}u^4 + \frac{2u^6}{5(1-|u|)}} \leq -\frac{1}{2u^2(1 + \frac{1}{3}u^2 + \frac{2}{5}u^4)}$$

since $(1-|u|)^{-1} \leq 2$ for u in this range. Accordingly,

$$\frac{1}{u \ln\left(\frac{1-u}{1+u}\right)} \leq -\frac{1}{2u^2}\left(1 - \frac{u^2}{3} - \frac{2u^4}{5}\right), \qquad 0 < |u| < \frac{1}{2}.$$

By taking $u = T\kappa^*(s)/2$, we thus obtain from (4.13)

$$k \leq \frac{ST^3}{3} + \frac{T^5}{40} \int_0^S [\kappa^*(s)]^2 \, ds \left(T \leq \min \frac{1}{|\kappa^*(s)|}, \text{ unless } \kappa^* \equiv 0\right) \tag{4.14}$$

which is the desired upper bound. As one might expect, (4.14) reduces to (4.3) for $\kappa^* = 0$.

In the foregoing derivation of (4.14), we emphasized mathematical precision, perhaps at the expense of obscuring the idea used to find the function $\boldsymbol{\psi}$ that we used. We were guided by the example of the rectangle to seek $\boldsymbol{\psi}$ in the form

$$\boldsymbol{\psi} = \psi(s, \lambda)\mathbf{n}^*(s).$$

We then integrated

$$\nabla \cdot \boldsymbol{\psi} = \frac{1}{1 - \lambda\kappa^*(s)} \frac{\partial}{\partial\lambda} [(1 - \lambda\kappa^*(s))\psi] = -2$$

with respect to λ to get

$$\psi = \frac{1 - \lambda\kappa^*(s)}{\kappa^*(s)} + \frac{c(s)}{1 - \lambda\kappa^*(s)}.$$

Finally, we made a choice of c that renders the variation of the integral

$$I(c) = \int_{\mathscr{D}} \psi^2 \, dA$$

zero, namely

$$c(s) = \frac{T}{\ln\left(\frac{1 - T\kappa^*(s)/2}{1 + T\kappa^*(s)/2}\right)}.$$

As for lower bounds, no derivation is required, for Pólya[8] has concluded that

$$k > \tfrac{4}{3}A^3 l^{-2}.$$

Since

$$A = ST, \qquad l = 2(S + T), \tag{4.15}$$

this bound may be written as

$$k > \frac{ST^3}{3}\left(1 + \frac{T}{S}\right)^{-2}. \tag{4.16}$$

5. BOUNDS ON t

Our aim is to apply (2.8), taking $\bar{\tau}$, $\underline{\tau}$ from (3.1), (3.5), i.e.

$$\bar{\tau} = \rho(2 - \kappa\rho), \qquad \underline{\tau} = \frac{2A}{l}, \tag{5.1}$$

and $\bar{k}$, $\underline{k}$ from (4.14), (4.16). For the class of strip domains introduced in Section 4,

$$\rho = T/2, \qquad \kappa = -\frac{\hat{\kappa}}{1 - \hat{\kappa}T/2}, \tag{5.2}$$

where

$$\hat{\kappa} = \max|\kappa^*(s)| \qquad (0 \le s \le S). \tag{5.3}$$

Thus, by (5.1), (4.15),

$$\bar{\tau} = T + \frac{\hat{\kappa}T^2}{2(2 - \hat{\kappa}T)}, \qquad \underline{\tau} = \frac{ST}{S + T}. \tag{5.4}$$

Accordingly, (2.7), (2.8), (4.14), and (4.16) now furnish

$$\frac{3}{ST^2}\frac{1}{\left(1 + \frac{T}{S}\right)\left(1 + \frac{3T^2}{40S}\int_0^S [\kappa^*(s)]^2\,\mathrm{d}s\right)} \le \frac{t}{m} < \frac{3}{ST^2}\left[1 + \frac{\hat{\kappa}T}{2(2 - \hat{\kappa}T)}\right]\left(1 + \frac{T}{S}\right)^2. \tag{5.5}$$

Let

$$\sigma = \frac{3m}{ST^2}, \tag{5.6}$$

and use the inequality

$$(1 + u)^{-1} \ge 1 - u \qquad (u \ge 0)$$

in the left hand member of (5.5) to get

$$\left(1 - \frac{T}{S}\right)\left(1 - \frac{3T^2}{40S}\int_0^S [\kappa^*(s)]^2\,\mathrm{d}s\right) \le \frac{t}{\sigma} < \left(1 + \frac{T}{S}\right)^2\left[1 + \frac{\hat{\kappa}T}{2(2 - \hat{\kappa}T)}\right]. \tag{5.7}$$

Finally, since

$$\int_0^S [\kappa^*(s)]^2\,\mathrm{d}s \le S\hat{\kappa}^2,$$

we arrive at

$$\left(1-\frac{T}{S}\right)\left(1-\frac{3}{40}\hat{\kappa}^2T^2\right) \leq \frac{t}{\sigma} < \left(1+\frac{T}{S}\right)^2\left[1+\frac{\hat{\kappa}T}{2(2-\hat{\kappa}T)}\right]. \tag{5.8}$$

From these inequalities, it is at once evident that

$$\frac{t}{\sigma} \to 1 \quad \text{as} \quad T \to 0,$$

provided $\hat{\kappa}$, S are held fixed. In addition, (5.8) enables one to decide how small T needs to be in comparison to S and $\hat{\kappa}^{-1}$ in order to approximate t by σ to within a prescribed error.

As an example, consider the case of a split circular tube of mean radius R and thickness T. In this case,

$$\hat{\kappa} = \frac{1}{R}, \qquad S = 2\pi R,$$

and as a consequence (5.5), (5.6) furnish

$$\underline{f}(\varepsilon) \overset{\text{def}}{\equiv} \frac{1}{\left(1+\frac{\varepsilon}{2\pi}\right)\left(1+\frac{3}{40}\varepsilon^2\right)} \leq \frac{t}{\sigma} < \left(1+\frac{\varepsilon}{2\pi}\right)^2\left[1+\frac{\varepsilon}{2(2-\varepsilon)}\right] \overset{\text{def}}{\equiv} \bar{f}(\varepsilon),$$

where $\varepsilon = T/R$. A few values of $\underline{f}$ and $\bar{f}$ are given in Table 1.

Table 1

ε	$\underline{f}$	$\bar{f}$
0·01	0·9984	1·005
0·05	0·9919	1·029
0·10	0·9835	1·059
0·20	0·9662	1·123
0·30	0·9480	1·194

Acknowledgement—We are indebted to the National Science Foundation for the support provided by NSF Grant GK-31070. We would also like to express our gratitude to Professor L. E. Payne of Cornell University for calling our attention to the likelihood that (3.1) would be useful in connection with thin-strip cross sections. Finally, we would like to extend our thanks to Professor M. J. Turteltaub, Central University of Venezuela, for going over an earlier version of this paper.

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Абстракт—Определяются пределы для максимального суммарного напряжения сдвига. Далее, они используются для исследования приближенного выражения, фигурующего в величине, предсказанной для тонких сечений на основе мембранной аналогии.